

Properties of the Exotic Characteristic Homomorphism for a Pair of Lie Algebroids, Relationship with the Koszul Homomorphism for a Pair of Lie algebras

by

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Abstract

We examine functorial and homotopy properties of the exotic characteristic homomorphism in the category of Lie algebroids which was lastly obtained by the authors in [4]. This homomorphism depends on a triple (A, B, ∇) where $B \subset A$ are regular Lie algebroids, both over the same regular foliated manifold (M, F) , and ∇ is a flat L -connection in A , where L is an arbitrary Lie algebroid over M . The Rigidity Theorem (i.e. the independence from the choice of homotopic Lie subalgebroids of B) is obtained. The exotic characteristic homomorphism is factorized by one (called universal) obtained for a pair of regular Lie algebroids. We raise the issue of injectivity of the universal homomorphism and establish injectivity for special cases. Here the Koszul homomorphism for pairs of isotropy Lie algebras plays a major role.

1 Introduction

In [4] we constructed some secondary (exotic) characteristic homomorphism

$$\Delta_{(A,B,\nabla)\#} : H^\bullet(\mathfrak{g}, B) \longrightarrow H^\bullet(L)$$

for a triple (A, B, ∇) , in which A is a regular Lie algebroid over a foliated manifold (M, F) , B its regular subalgebroid on the same foliated manifold (M, F) , \mathfrak{g} the kernel of the anchor of A , and $\nabla : L \rightarrow A$ a flat L -connection in A for an arbitrary Lie algebroid L over M . The domain of this homomorphism is the Lie algebroid analog to the relative cohomology algebra for a pair of Lie algebras defined in [5]. $\Delta_{(A,B,\nabla)\#}$ generalizes some known secondary characteristic classes: for flat principal fibre bundles with a reduction (Kamber, Tondeur [13]) and two approaches to flat characteristic classes for Lie algebroids, the one for regular Lie algebroids due to Kubarski [19] and the one for representations of not necessarily regular Lie algebroids on vector bundles developed by Crainic ([6], [7]).

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For $L = A$ and $\nabla = \text{id}_A$ we obtain a new universal characteristic homomorphism $\Delta_{(A,B)\#}$, which factorizes the characteristic homomorphism $\Delta_{(A,B,\nabla)\#}$ for each flat L -connection $\nabla : L \rightarrow A$, i.e.

$$\Delta_{(A,B,\nabla)\#} = \nabla^\# \circ \Delta_{(A,B)\#}. \quad (1.1)$$

Clearly, no class from the kernel of $\Delta_{(A,B)\#}$ is an obstruction to the fact that the given flat connection $\nabla : L \rightarrow A$ is induced by a connection in B . By this reason, we put the following (new) question for secondary characteristic classes: **Is the exotic universal characteristic homomorphism $\Delta_{(A,B)\#}$ a monomorphism?** In Section 4, we give the positive answer under some assumptions.

We remark that the characteristic homomorphism $\Delta_{(A,B)\#} : H^\bullet(\mathfrak{g}, B) \rightarrow H^\bullet(A)$ really depends only on the inclusion $i : B \hookrightarrow A$, [4]. For this inclusion — as for any homomorphism of Lie algebroids — I. Vaisman in [22] defined secondary characteristic classes $\mu_{2h-1}(i)$ lying in $H^{4h-3}(B)$, i.e. in a different group of cohomology than universal characteristic classes (which belong to $H^\bullet(A)$). The detailed relationships between these frameworks for secondary characteristic classes will be the subject of the next paper. We point out only (see [9], [6], [7], [4]) that the modular class $\text{mod}(\tilde{A})$ of a Lie algebroid \tilde{A} (for a definition see [23], [8]) — which is equal to the first secondary characteristic class of the anchor of \tilde{A} — in case where the basic connection $\nabla = (\hat{\nabla}, \check{\nabla})$ given in [9] is flat, can express in the term of secondary characteristic homomorphism $\Delta_{(A,B,\nabla)\#}$ for the triple (A, B, ∇) where $A = \mathcal{A}(\tilde{A} \oplus T^*M)$ is a Lie algebroid of the vector bundle $\tilde{A} \oplus T^*M$, $B = \mathcal{A}(\tilde{A} \oplus T^*M, \{h\})$ is its Lie subalgebroid being the Lie algebroid of the Riemannian reduction $(\tilde{A} \oplus T^*M, \{h\})$ and $\nabla = (\hat{\nabla}, \check{\nabla})$; namely $\text{mod}(\tilde{A})$ and $\Delta_{(A,B,\nabla)\#}(y_1)$ are equal up to a constant. The first secondary characteristic class $\mu_1(i)$ of the considered inclusion $i : B \hookrightarrow A$ equals $\text{mod}(B) - i^\#(\text{mod } A)$. So, it can be expressed then in terms of characteristic classes from the images of suitable characteristic homomorphisms of the form $\Delta_{(A',B',\nabla')\#}$ constructed in [4].

The meaning of the classical exotic characteristic homomorphism for a principal bundle with a given reduction consist in that it measures the incompatibility of two geometric structures on a given principal bundle: its reduction and a flat connection. Namely, if a flat connection is a connection in a given reduction, this exotic characteristic homomorphism is trivial (i.e. it is the zero homomorphism in all positive degrees). The exotic characteristic homomorphism $\Delta_{(A,B,\nabla)\#}$ for Lie algebroids has the similar meaning.

The classical exotic homomorphism for given principal bundle P and its reduction P' has stronger property: it is trivial if a given flat connection has values in any reduction homotopic to P' (in some cases every two H -reduction are homotopic [13]).

Chapter 3 concerns investigation of homotopic properties of the generalized exotic homomorphism $\Delta_{(A,B,\nabla)\#}$ for Lie algebroids. We examine here the notion of homotopic Lie subalgebroids, which was introduced in [19] as a natural generalization of the notion of homotopic two H -reductions of a principal bundle. We show also functorial properties of the considered homomorphism $\Delta_{(A,B,\nabla)\#}$ with respect to homomorphisms of Lie algebroids (not necessary over identity on the base manifold).

Chapter 4 concerns exotic universal characteristic homomorphism $\Delta_{(A,B)\#}$ in two cases. First, for the trivial case of Lie algebroids over a point, i.e. for Lie algebras. This universal homomorphism is, in fact, equivalent (up to the sign) to the well known "Koszul homomorphism" $\Delta_{(\mathfrak{g},\mathfrak{h})\#} : H^\bullet(\mathfrak{g}/\mathfrak{h}) \rightarrow H^\bullet(\mathfrak{g})$ for a pair of Lie algebras $(\mathfrak{g}, \mathfrak{h})$, $\mathfrak{h} \subset \mathfrak{g}$ [14], [11]. In [11] the injectivity of $\Delta_{(\mathfrak{g},\mathfrak{h})\#}$ is considered and used to investigate of the cohomology algebra of the homogeneous manifolds G/H . Next, applying the Lie functor for principal fibre bundles it gives a new universal homomorphism for the reduction of a

principal fibre bundle. It factorizes the standard secondary characteristic homomorphisms $\Delta_{(P,P',\omega)\#}$ for any flat connections ω in P . In Section 5, using functorial properties of the inclusion $\iota_x : (\mathbf{g}_x, \mathbf{h}_x) \rightarrow (A, B)$ over the map $\{x\} \hookrightarrow M$, where $\mathbf{g}_x, \mathbf{h}_x$ are isotropy algebras of Lie algebroids A and B at $x \in M$, we show connection of the exotic universal characteristic homomorphism $\Delta_{(A,B)\#}$ with the Koszul homomorphism for isotropy Lie algebras $(\mathbf{g}_x, \mathbf{h}_x)$. We find some conditions under which the characteristic homomorphism $\Delta_{(A,B)\#}$, for a pair $B \subset A$, is a monomorphism. The presented considerations show that the Koszul homomorphism plays an essential role for the study of exotic characteristic classes.

In the paper we suppose that the reader is familiar with Lie algebroids and for more about Lie algebroids and its connections we refer to [21], [12], [20], [3], [9].

2 Construction of Exotic Characteristic Homomorphism

We shall briefly explain the construction of the exotic characteristic homomorphism and the universal exotic characteristic homomorphism on Lie algebroids from [4].

Let $(A, [\cdot, \cdot], \#_A)$ be a regular Lie algebroid over a foliated manifold (M, F) , B its regular **subalgebroid** on the same foliated manifold (M, F) , L a Lie algebroid over M and $\nabla : L \rightarrow A$ a **flat** L -connection in A . We call the triple

$$(A, B, \nabla)$$

an *FS-Lie algebroid*. Let $\lambda : F \rightarrow B$ be an arbitrary connection in B . Then $j \circ \lambda : F \rightarrow A$ is a connection in A . Let $\check{\lambda} : A \rightarrow \mathbf{g}$ be its connection form. Summarizing, we have a flat L -connection $\nabla : L \rightarrow A$ in A and the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbf{g} & \xhookrightarrow{\iota} & A & \xrightarrow{\#_A} & F \longrightarrow 0 \\ & & \uparrow & & \uparrow j & & \parallel \\ 0 & \longrightarrow & \mathbf{h} & \longrightarrow & B & \xrightleftharpoons[\lambda]{\#_B} & F \longrightarrow 0 \end{array}$$

The homomorphism $\omega_{B,\nabla} : L \rightarrow \mathbf{g}/\mathbf{h}$, $\omega_{B,\nabla}(w) = [-\check{\lambda} \circ \nabla(w)]$ does not depend on the choice of an auxiliary connection $\lambda : F \rightarrow B$ and $\omega_{B,\nabla} = 0$ if ∇ takes values in B . Let us define a homomorphism of algebras

$$\Delta_{(A,B,\nabla)} : \Gamma\left(\bigwedge^k (\mathbf{g}/\mathbf{h})^*\right) \longrightarrow \Omega(L), \quad (2.1)$$

$$(\Delta_{(A,B,\nabla)} \Psi)_x (w_1 \wedge \dots \wedge w_k) = \langle \Psi_x, \omega_{B,\nabla}(w_1) \wedge \dots \wedge \omega_{B,\nabla}(w_k) \rangle, \quad w_i \in L|_x.$$

In the algebra $\Gamma(\bigwedge (\mathbf{g}/\mathbf{h})^*)$ we distinguish the subalgebra $(\Gamma(\bigwedge (\mathbf{g}/\mathbf{h})^*))^{\Gamma(B)}$ of invariant cross-sections with respect to the representation of the Lie algebroid B in the vector bundle $\bigwedge (\mathbf{g}/\mathbf{h})^*$, associated to the adjoint one $\text{ad}_{B,\mathbf{h}} : B \rightarrow \text{A}(\mathbf{g}/\mathbf{h})$, $\text{ad}_{B,\mathbf{h}}(\xi)([\nu]) = [[\xi, \nu]]$, $\xi \in \Gamma(B)$, $\nu \in \Gamma(\mathbf{g})$. Recall, that $\Psi \in \left(\Gamma\left(\bigwedge^k (\mathbf{g}/\mathbf{h})^*\right)\right)^{\Gamma(B)}$ if and only if

$$(\#_B \circ \xi) \langle \Psi, [\nu_1] \wedge \dots \wedge [\nu_k] \rangle = \sum_{j=1}^k (-1)^{j-1} \langle \Psi, [[j \circ \xi, \nu_j]] \wedge [\nu_1] \wedge \dots \wedge \hat{j} \wedge \dots \wedge [\nu_k] \rangle$$

for all $\xi \in \Gamma(B)$ and $\nu_j \in \Gamma(\mathbf{g})$ (see [16]). In the space $(\Gamma(\bigwedge(\mathbf{g}/\mathbf{h})^*))^{\Gamma(B)}$ of invariant cross-sections there exists a differential operator $\bar{\delta}$ defined by

$$\langle \bar{\delta}\Psi, [\nu_1] \wedge \dots \wedge [\nu_k] \rangle = \sum_{i < j} (-1)^{i+j+1} \langle \Psi, [[\nu_i, \nu_j]] \wedge [\nu_1] \wedge \dots \hat{i} \dots \hat{j} \dots \wedge [\nu_k] \rangle,$$

(see [19]) and we obtain the cohomology algebra

$$\mathbf{H}^\bullet(\mathbf{g}, B) := \mathbf{H}^\bullet((\Gamma(\bigwedge(\mathbf{g}/\mathbf{h})^*))^{\Gamma(B)}, \bar{\delta}).$$

The homomorphism $\Delta_{(A,B,\nabla)}$ commutes with the differentials $\bar{\delta}$ and d_L , where d_L is the differential operator in $\Omega(L) = \Gamma(\bigwedge L^*)$, see [4]. In this way we obtain the cohomology homomorphism

$$\Delta_{(A,B,\nabla)\#} : \mathbf{H}^\bullet(\mathbf{g}, B) \longrightarrow \mathbf{H}^\bullet(L).$$

In the case where $L = A$ and $\nabla = \text{id}_A : A \rightarrow A$ is the identity map, we have particular case of a homomorphism for the pair (A, B) :

$$\begin{aligned} \Delta_{(A,B)} &:= \Delta_{(A,B,\text{id}_A)} : \Gamma(\bigwedge^k(\mathbf{g}/\mathbf{h})^*)^{\Gamma(B)} \longrightarrow \Omega(A), \\ (\Delta_{(A,B)}\Psi)_x(v_1 \wedge \dots \wedge v_k) &= \langle \Psi_x, [-\check{\lambda}(v_1)] \wedge \dots \wedge [-\check{\lambda}(v_k)] \rangle, \quad v_i \in A|_x. \end{aligned}$$

$\Delta_{(A,B,\nabla)}$ can be written as a composition

$$\Delta_{(A,B,\nabla)} : \Gamma(\bigwedge(\mathbf{g}/\mathbf{h})^*) \xrightarrow{\Delta_{(A,B)}} \Omega(A) \xrightarrow{\nabla^*} \Omega(L),$$

where ∇^* is the pullback of forms. For this reason, $\Delta_{(A,B)}$ induces the cohomology homomorphism

$$\Delta_{(A,B)\#} : \mathbf{H}^\bullet(\mathbf{g}, B) \longrightarrow \mathbf{H}^\bullet(A),$$

which factorizes $\Delta_{(A,B,\nabla)\#}$ for every flat L -connection $\nabla : L \rightarrow A$:

$$\Delta_{(A,B,\nabla)\#} : \mathbf{H}^\bullet(\mathbf{g}, B) \xrightarrow{\Delta_{(A,B)\#}} \mathbf{H}^\bullet(A) \xrightarrow{\nabla^\#} \mathbf{H}^\bullet(L). \quad (2.2)$$

The map $\Delta_{(A,B,\nabla)\#}$ is called the *characteristic homomorphism* of the FS-Lie algebroid (A, B, ∇) . We call elements of a subalgebra $\text{Im } \Delta_{(A,B,\nabla)\#} \subset \mathbf{H}^\bullet(L)$ the *secondary (exotic) characteristic classes* of this algebroid. In particular, $\Delta_{(A,B)\#} = \Delta_{(A,B,\text{id}_A)\#}$ is the characteristic homomorphism of the Lie subalgebroid $B \subset A$, which we call the *universal exotic characteristic homomorphism*; the characteristic classes from its image we call the *universal characteristic classes* of the pair $B \subset A$.

The secondary characteristic homomorphism for FS-Lie algebroids generalizes the following known characteristic classes: for flat regular Lie algebroids (Kubarski), for flat principal fibre bundles with a reduction (Kamber, Tondeur) and for representations of Lie algebroids on vector bundles (Crainic).

1. For $L = F$ we obtain the case in which $\nabla : F \rightarrow A$ is a usual connection in A . In this way the exotic characteristic homomorphism is a generalization of one for a flat regular Lie algebroid given in [19], see [4].
2. For $L = TM$ and $A = TP/G$, and $B = TP'/H$ (P' is an H -reduction of P) we obtain the case equivalent to the standard classical theory on principal fibre bundles [13] (see [4] and Section 4.2 below for more details).

3. [4] Let $A = \mathcal{A}(\mathfrak{f})$ be the Lie algebroid of a vector bundle \mathfrak{f} over a manifold M , $B = \mathcal{A}(\mathfrak{f}, \{h\}) \subset A$ its Riemannian reduction ([17]), L a Lie algebroid over M , $\nabla : L \rightarrow \mathcal{A}(\mathfrak{f})$ an L -connection on \mathfrak{f} . Let $\Delta_{\#}$ denote the exotic characteristic homomorphism for FS-Lie algebroid $(\mathcal{A}(\mathfrak{f}), \mathcal{A}(\mathfrak{f}, \{h\}), \nabla)$. If the vector bundle \mathfrak{f} is nonorientable or orientable and of odd rank n , then the domain of $\Delta_{\#}$ is isomorphic with $\bigwedge(y_1, y_3, \dots, y_{n'})$ where n' is the largest odd integer $\leq n$ and $y_{2k-1} \in H^{4k-3}(\text{End } \mathfrak{f}, \mathcal{A}(\mathfrak{f}, \{h\}))$ is represented by the multilinear trace form $\tilde{y}_{2k-1} \in \Gamma\left(\bigwedge^{4k-3}(\text{End } \mathfrak{f}/\text{Sk } \mathfrak{f})^*\right)$. Then the image of $\Delta_{\#}$ is generated by the Crainic classes $u_1(\mathfrak{f}), u_5(\mathfrak{f}), \dots, u_{4\lfloor \frac{n+3}{4} \rfloor - 3}(\mathfrak{f})$ (for details about the classes developed by Crainic see [6], [7], [4]). If \mathfrak{f} is orientable of even rank $n = 2m$ with a volume form v , the domain of $\Delta_{\#}$ is additionally generated by some class $y_{2m} \in H^{2m}(\text{End } \mathfrak{f}, \mathcal{A}(\mathfrak{f}, \{h, v\}))$ represented by a form induced by the Pfaffian and where $\mathcal{A}(\mathfrak{f}, \{h, v\})$ is the Lie algebroid of the $SO(n, \mathbb{R})$ -reduction $\mathcal{L}(\mathfrak{f}, \{h, v\})$ of the frames bundle $\mathcal{L}\mathfrak{f}$ of \mathfrak{f} ; see [4]. Then the algebra of exotic characteristic classes for $(\mathcal{A}(\mathfrak{f}), \mathcal{A}(\mathfrak{f}, \{h, v\}), \nabla)$ is generated by $u_1(\mathfrak{f}), u_5(\mathfrak{f}), \dots, u_{4\lfloor \frac{n+3}{4} \rfloor - 3}(\mathfrak{f})$ and additionally by $\Delta_{\#}(y_{2m})$. In [4] we give an example of FS-Lie algebroid where the Pfaffian induces the non-zero characteristic class.

From (2.2) one can see that for a pair of regular Lie algebroids (A, B) , $B \subset A$, both over a foliated manifold (M, F) , and for an arbitrary element $\zeta \in H^{\bullet}(\mathfrak{g}, B)$ there exists a (universal) cohomology class $\Delta_{(A,B)\#}(\zeta) \in H^{\bullet}(A)$ such that for any Lie algebroid L over M and a flat L -connection $\nabla : L \rightarrow A$ the equality

$$\Delta_{(A,B,\nabla)\#}(\zeta) = \nabla^{\#}(\Delta_{(A,B)\#}(\zeta))$$

holds. Therefore, no element from the kernel of $\Delta_{(A,B)\#}$ can be used to compare the flat connection ∇ with a reduction $B \subset A$. Hence it is interesting the following

Problem 2.1 *Is the characteristic homomorphism $\Delta_{(A,B)\#} : H^{\bullet}(\mathfrak{g}, B) \rightarrow H^{\bullet}(A)$ a monomorphism for a given $B \subset A$? The answer yes holds in some cases, see below.*

3 Functoriality and Homotopic Properties

3.1 Functoriality

Let (A, B) and (A', B') be two pairs of regular Lie algebroids over (M, F) and (M', F') , respectively, where $B \subset A$, $B' \subset A'$, and let $H : A' \rightarrow A$ be a homomorphism of Lie algebroids over a mapping $f : (M', F') \rightarrow (M, F)$ of foliated manifolds such that $H[B'] \subset B$. We write $(H, f) : (A', B') \rightarrow (A, B)$. Let $H^{+\#} : H^{\bullet}(\mathfrak{g}, B) \rightarrow H^{\bullet}(\mathfrak{g}', B')$ be the homomorphism of cohomology algebras induced by the pullback $H^{+*} : \Gamma\left(\bigwedge^k(\mathfrak{g}/\mathfrak{h})^*\right) \rightarrow \Gamma\left(\bigwedge^k(\mathfrak{g}'/\mathfrak{h}')^*\right)$, see [19, Proposition 4.2].

Theorem 3.1 (The functoriality of $\Delta_{(A,B)\#}$) *For a given pair of regular Lie algebroids (A, B) , (A', B') and a homomorphism $(H, f) : (A', B') \rightarrow (A, B)$ we have the commutativity of the following diagram*

$$\begin{array}{ccc} H^{\bullet}(\mathfrak{g}, B) & \xrightarrow{\Delta_{(A,B)\#}} & H^{\bullet}(A) \\ H^{+\#} \downarrow & & \downarrow H^{\#} \\ H^{\bullet}(\mathfrak{g}', B') & \xrightarrow{\Delta_{(A',B')\#}} & H^{\bullet}(A'). \end{array}$$

Proof. One can see that $H^+ \circ \check{\lambda}'(u') - \check{\lambda}(Hu') \in \mathbf{h}$ for all $u' \in A'$, where λ and λ' are auxiliary connections in B and B' , respectively. Applying this fact, it is sufficient to check the commutativity of the diagram on the level of forms. The calculations are left to the reader. ■

Definition 3.2 Let (A', B', ∇') and (A, B, ∇) be two FS-Lie algebroids on foliated manifolds (M', F') and (M, F) , respectively, where $\nabla : L \rightarrow A$ and $\nabla' : L' \rightarrow A'$ are flat connections. By a *homomorphism*

$$H : (A', B', \nabla') \longrightarrow (A, B, \nabla)$$

over $f : (M', F') \rightarrow (M, F)$ we mean a pair (H, h) such that:

- $H : A' \rightarrow A$ is a homomorphism of regular Lie algebroids over f and $H[B'] \subset B$,
- $h : L' \rightarrow L$ is also a homomorphism of Lie algebroids over f ,
- $\nabla \circ h = H \circ \nabla'$.

Clearly, $h^\# \circ \nabla'^\# = \nabla^\# \circ H^\#$. So, from (2.2) and Theorem 3.1 we obtain as a corollary the following theorem:

Theorem 3.3 (The functoriality of $\Delta_{(A,B,\nabla)^\#}$) *The following diagram*

$$\begin{array}{ccc} \mathbf{H}^\bullet(g, B) & \xrightarrow{\Delta_{(A,B,\nabla)^\#}} & \mathbf{H}^\bullet(L) \\ \downarrow H^\# & & \downarrow h^\# \\ \mathbf{H}^\bullet(g', B') & \xrightarrow{\Delta_{(A',B',\nabla')^\#}} & \mathbf{H}^\bullet(L') \end{array}$$

commutes.

3.2 Homotopy Invariance

We recall the definition of homotopy between homomorphisms of Lie algebroids.

Definition 3.4 [18] Let $H_0, H_1 : L' \rightarrow L$ be two homomorphisms of Lie algebroids. By a *homotopy joining H_0 to H_1* we mean a homomorphism of Lie algebroids

$$H : T\mathbb{R} \times L' \longrightarrow L,$$

such that $H(\theta_0, \cdot) = H_0$ and $H(\theta_1, \cdot) = H_1$, where θ_0 and θ_1 are null vectors tangent to \mathbb{R} at 0 and 1, respectively. We say that H_0 and H_1 are *homotopic* and write $H_0 \sim H_1$. We say that $F : L' \rightarrow L$ is a *homotopy equivalence* if there is a homomorphism $G : L \rightarrow L'$ such that $G \circ F \sim \text{id}_{L'}$ and $F \circ G \sim \text{id}_L$.

The homotopy $H : T\mathbb{R} \times L' \rightarrow L$ determines a chain homotopy operator ([18], [2]) which implies that $H_0^\# = H_1^\# : \mathbf{H}^\bullet(L) \rightarrow \mathbf{H}^\bullet(L')$.

Definition 3.5 [19] Two Lie subalgebroids $B_0, B_1 \subset A$ (both over the same foliated manifold (M, F)) are said to be *homotopic*, if there exists a Lie subalgebroid $B \subset T\mathbb{R} \times A$ over $(\mathbb{R} \times M, T\mathbb{R} \times F)$, such that for $t \in \{0, 1\}$

$$v_x \in B_{t|x} \text{ if and only if } (\theta_t, v_x) \in B_{|(t,x)}. \quad (3.1)$$

B is called a *subalgebroid joining* B_0 with B_1 .

See [19, Proposition 5.2] to compare the relation of homotopic subbundles of a principal bundle with the relation of homotopic subalgebroids.

Let B_0, B_1 be two homotopic Lie subalgebroids over (M, F) and let $B \subset T\mathbb{R} \times A$ be a subalgebroid of $T\mathbb{R} \times A$ joining B_0 with B_1 , $t \in \{0, 1\}$. For the homomorphism of Lie algebroids $F_t^A : A \rightarrow T\mathbb{R} \times A$, $v_x \mapsto (\theta_t, v_x)$ over $f_t : M \rightarrow \mathbb{R} \times M$, $f_t(x) = (t, x)$, (3.1) yields $F_t^A[B_t] \subset B$. Applying the functoriality of $\Delta_{t\#} := \Delta_{(A, B_t)\#}$ and $\Delta_{(A, B)\#}$ (see Theorem 3.1), we obtain the commutativity of the diagram

$$\begin{array}{ccccc} & & \Delta_{0\#} & & \\ & \text{H}^\bullet(\mathfrak{g}, B_0) & \longrightarrow & \text{H}^\bullet(A) & \\ & \uparrow F_0^{+\#} \simeq & & \uparrow F_0^{A\#} & \\ \alpha & \text{H}^\bullet(0 \times \mathfrak{g}, B) & \xrightarrow{\Delta_{(T\mathbb{R} \times A, B)\#}} & \text{H}^\bullet(T\mathbb{R} \times A) & \\ & \downarrow F_1^{+\#} \simeq & & \downarrow F_1^{A\#} & \\ & \text{H}^\bullet(\mathfrak{g}, B_1) & \xrightarrow{\Delta_{1\#}} & \text{H}^\bullet(A) & \end{array}$$

where $F_t^{+\#} \equiv (F_t^A)^{+\#}$. In the paper [19] it is shown that $F_t^{+\#}$ are isomorphisms of algebras. We add that in the proof of this fact one makes use of some theorem concerning invariant cross-sections over $\mathbb{R} \times M$ (with respect to a suitable representation) and one uses global solutions of some system of first-order partial differential equations with parameters, see [20].

For any flat L -connection $\nabla : L \rightarrow A$, the induced $T\mathbb{R} \times L$ -connection $\text{id}_{T\mathbb{R}} \times \nabla$ is also flat. F_t^A determines a homomorphism

$$(A, B_t, \nabla) \longrightarrow (T\mathbb{R} \times A, B, \text{id}_{T\mathbb{R}} \times \nabla)$$

of FS-Lie algebroids over $f_t : M \rightarrow \mathbb{R} \times M$; and so we can complete the previous diagram to the following one.

$$\begin{array}{ccccccc} & & \Delta_{0\#} & & \nabla^\# & & \\ & \text{H}^\bullet(\mathfrak{g}, B_0) & \longrightarrow & \text{H}^\bullet(A) & \longrightarrow & \text{H}^\bullet(L) & \\ & \uparrow F_0^{+\#} \simeq & & \uparrow F_0^{A\#} & & \uparrow F_0^{L\#} & \\ \alpha & \text{H}^\bullet(0 \times \mathfrak{g}, B) & \xrightarrow{\Delta_{(T\mathbb{R} \times A, B)\#}} & \text{H}^\bullet(T\mathbb{R} \times A) & \xrightarrow{(\text{id} \times \nabla)^\#} & \text{H}^\bullet(T\mathbb{R} \times L) & \\ & \downarrow F_1^{+\#} \simeq & & \downarrow F_1^{A\#} & & \downarrow F_1^{L\#} & \\ & \text{H}^\bullet(\mathfrak{g}, B_1) & \xrightarrow{\Delta_{1\#}} & \text{H}^\bullet(A) & \xrightarrow{\nabla^\#} & \text{H}^\bullet(L) & \end{array}$$

Observe that the rows of the above diagram are characteristic homomorphisms of FS-Lie algebroids. Since $F_0^L, F_1^L : L \rightarrow T\mathbb{R} \times L$ are homotopic homomorphisms, then $F_0^{L\#} = F_1^{L\#}$. To prove the homotopy independence of the exotic characteristic homomorphism (in the sense of Definition 3.5, i.e., the independence of a class of homotopic subalgebroids), it is sufficient to show that $F_0^{L\#}, F_1^{L\#}$ are isomorphisms.

We shall see below that F_t^L are homotopically equivalent. Take the projection $\pi : T\mathbb{R} \times L \rightarrow L$ (over $\text{pr}_2 : \mathbb{R} \times M \rightarrow M$). Of course, π is a homomorphism of Lie algebroids. Note that $F_{t_o}^L \circ \pi = (\hat{t}_o)_* \times \text{id}_L$, where $\hat{t}_o : \mathbb{R} \rightarrow \mathbb{R}$ is defined by $t \mapsto t_o$. We take $\tau : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, $(s, t) \mapsto t_o + s(t - t_o)$. Since the differential $f_* : TM \rightarrow TN$ of any smooth mapping $f : M \rightarrow N$ is a homomorphism of Lie algebroids [18], we obtain that the map $\tau_* : T(\mathbb{R} \times \mathbb{R}) = T\mathbb{R} \times T\mathbb{R} \rightarrow T\mathbb{R}$ is a homomorphism of Lie algebroids. We put

$$\begin{aligned} H : T\mathbb{R} \times (T\mathbb{R} \times L) &\longrightarrow T\mathbb{R} \times L, \\ H &= \tau_* \times \text{id}_L. \end{aligned}$$

Since

$$\begin{aligned} H(\theta_0, \cdot, \cdot) &= \tau(0, \cdot)_* \times \text{id}_L = (\hat{t}_o)_* \times \text{id}_L = F_{t_o}^L \circ \pi, \\ H(\theta_1, \cdot, \cdot) &= \tau(\cdot, 1)_* \times \text{id}_L = \text{id}_{T\mathbb{R} \times L}, \end{aligned}$$

H is a homotopy joining $F_{t_o}^L \circ \pi$ with $\text{id}_{T\mathbb{R} \times L}$, i.e. $F_{t_o}^L \circ \pi \sim \text{id}_{T\mathbb{R} \times L}$. Evidently, $\pi \circ F_{t_o}^L = \text{id}_L$. Therefore, F_t^L are isomorphisms.

These facts lead us to the following result:

Theorem 3.6 (The Rigidity Theorem) *If $B_0, B_1 \subset A$ are homotopic subalgebroids of A and $\nabla : L \rightarrow A$ is a flat L -connection in A , characteristic homomorphisms $\Delta_{(A, B_0, \nabla)\#} : \mathbf{H}^\bullet(\mathfrak{g}, B_0) \rightarrow \mathbf{H}^\bullet(L)$ and $\Delta_{(A, B_1, \nabla)\#} : \mathbf{H}^\bullet(\mathfrak{g}, B_1) \rightarrow H_L(M)$ are equivalent in the sense that there exists an isomorphism of algebras*

$$\alpha : \mathbf{H}^\bullet(\mathfrak{g}, B_0) \xrightarrow{\sim} \mathbf{H}^\bullet(\mathfrak{g}, B_1)$$

such that

$$\Delta_{(A, B_1, \nabla)\#} \circ \alpha = \Delta_{(A, B_0, \nabla)\#}.$$

In particular, $\Delta_{(A, B_1)\#} \circ \alpha = \Delta_{(A, B_0)\#}$.

Corollary 3.7 *Let \mathfrak{f} be a vector bundle and $\mathcal{A}(\mathfrak{f})$ its Lie algebroid. Two Lie subalgebroids $B_0 = \mathcal{A}(\mathfrak{f}, \{h_0\})$, $B_1 = \mathcal{A}(\mathfrak{f}, \{h_1\})$ of the Lie algebroid $\mathcal{A}(\mathfrak{f})$, corresponding to two Riemannian metrics h_0, h_1 , are homotopic Lie subalgebroids [19]. Therefore, according to the Rigidity Theorem 3.6 we conclude that $\Delta_{(\mathcal{A}(\mathfrak{f}), \mathcal{A}(\mathfrak{f}, \{h_0\}))\#} = \Delta_{(\mathcal{A}(\mathfrak{f}), \mathcal{A}(\mathfrak{f}, \{h_1\}))\#}$, i.e. the characteristic homomorphism for the pair $(\mathcal{A}(\mathfrak{f}), \mathcal{A}(\mathfrak{f}, \{h\}))$ is an intrinsic notion for $\mathcal{A}(\mathfrak{f})$ not depending on the metric h .*

4 Particular Cases of the Universal Exotic Characteristic Homomorphism

4.1 The Koszul Homomorphism

In this section, we will consider the characteristic homomorphism $\Delta_{(\mathfrak{g}, \mathfrak{h})\#}$ for a pair of Lie algebras $(\mathfrak{g}, \mathfrak{h})$, $\mathfrak{h} \subset \mathfrak{g}$, and give a class of such pairs for which $\Delta_{(\mathfrak{g}, \mathfrak{h})\#}$ is a monomorphism.

An arbitrary Lie algebra is a Lie algebroid over a point with the zero map as an anchor. Considering the homomorphism of pairs of Lie algebras $(\text{id}_{\mathfrak{g}}, 0) : (\mathfrak{g}, 0) \rightarrow (\mathfrak{g}, \mathfrak{h})$, $\mathfrak{h} \subset \mathfrak{g}$, the functoriality property gives that

$$\Delta_{(\mathfrak{g}, \mathfrak{h})\#} = \Delta_{(\mathfrak{g}, 0)\#} \circ (\text{id}_{\mathfrak{g}}, 0)^{+\#} = -(\text{id}_{\mathfrak{g}}, 0)^{+\#} : H^\bullet(\mathfrak{g}, \mathfrak{h}) \rightarrow H^\bullet(\mathfrak{g})$$

is induced (in cohomology) by minus of the projection $s : \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{h}$, since $\Delta_{(\mathfrak{g}, 0)\#} : H^\bullet(\mathfrak{g}, 0) = H^\bullet(\mathfrak{g}) \xrightarrow{(-\text{id}_{\mathfrak{g}})\#} H^\bullet(\mathfrak{g})$. More precisely, if $k : (\bigwedge \mathfrak{g}^*)_{i_{\mathfrak{h}}=0, \theta_{\mathfrak{h}}=0} \rightarrow \bigwedge \mathfrak{g}^*$ denotes the inclusion from the basic subalgebra $(\bigwedge \mathfrak{g}^*)_{i_{\mathfrak{h}}=0, \theta_{\mathfrak{h}}=0}$ (i.e. subalgebra of invariant and horizontal elements of $\bigwedge \mathfrak{g}^*$ with respect to the Lie subalgebra \mathfrak{h}) to $\bigwedge \mathfrak{g}^*$ (see [11, p. 412]), then the secondary characteristic homomorphism $\Delta_{(\mathfrak{g}, \mathfrak{h})\#}$ for the pair $(\mathfrak{g}, \mathfrak{h})$ can be written as a superposition

$$\Delta_{(\mathfrak{g}, \mathfrak{h})\#} : H^\bullet(\mathfrak{g}, \mathfrak{h}) \xrightarrow[\cong]{(-s)\#} H^\bullet(\mathfrak{g}/\mathfrak{h}) \xrightarrow{k\#} H^\bullet(\mathfrak{g}),$$

where $H^\bullet(\mathfrak{g}/\mathfrak{h})$ denotes the cohomology algebra $H^\bullet((\bigwedge \mathfrak{g}^*)_{i_{\mathfrak{h}}=0, \theta_{\mathfrak{h}}=0}, d_{\mathfrak{g}})$.

Example 4.1 Let $(\mathfrak{g}, \mathfrak{h})$ be a reductive pair of Lie algebras ($\mathfrak{h} \subset \mathfrak{g}$), $s : \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{h}$ the canonical projection. Theorems IX and X from [11, sections 10.18, 10.19] yield that $k\#$ is injective if and only if $H^\bullet(\mathfrak{g}/\mathfrak{h})$ is generated by 1 and odd-degree elements. Therefore, because of $(-s)\#$ is an isomorphism of algebras, it follows that $\Delta_{(\mathfrak{g}, \mathfrak{h})\#}$ is injective if and only if $H^\bullet(\mathfrak{g}, \mathfrak{h})$ is generated by 1 and odd-degree elements. In a wide class of pairs of Lie algebras $(\mathfrak{g}, \mathfrak{h})$ such that \mathfrak{h} is reductive in \mathfrak{g} , the homomorphism $k\#$ is injective if and only if \mathfrak{h} is noncohomologous to zero (briefly: n.c.z.) in \mathfrak{g} (i.e. if the homomorphism of algebras $H^\bullet(\mathfrak{g}) \rightarrow H^\bullet(\mathfrak{h})$ induced by the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ is surjective). Tables I, II and III at the end of Section XI [11] contain many n.c.z. pairs, eg: $(\mathfrak{gl}(n), \mathfrak{so}(n))$ for odd n , $(\mathfrak{so}(n, \mathbb{C}), \mathfrak{so}(k, \mathbb{C}))$ for $k < n$, $(\mathfrak{so}(2m+1), \mathfrak{so}(2k+1))$ and $(\mathfrak{so}(2m), \mathfrak{so}(2k+1))$ for $k < m$ and others.

In view of the above, the examples below yield that the secondary characteristic homomorphism for the reductive pair $(\text{End}(V), \text{Sk}(V))$ of Lie algebras is a monomorphism for any odd dimensional vector space V and not a monomorphism for even dimensional.

Example 4.2 (The pair of Lie algebras $(\text{End}(V), \text{Sk}(V))$) (a) Let V be a vector space of odd dimension, $\dim V = 2m - 1$. Then

$$H^\bullet(\text{End}(V), \text{Sk}(V)) \cong H^\bullet(\mathfrak{gl}(2m-1, \mathbb{R}), O(2m-1)) \cong \bigwedge (y_1, y_3, \dots, y_{2m-1}),$$

where $y_{2k-1} \in H^{4k-3}(\text{End}(V), \text{Sk}(V))$ are represented by the multilinear trace forms ([10], [13]). We conclude from the previous example that $\Delta_{(\text{End}(V), \text{Sk}(V))\#}$ is injective.

(b) In the case where V is an even dimensional vector space ($\dim V = 2m$), we have [10]

$$H^\bullet(\text{End}(V), \text{Sk}(V)) \cong H^\bullet(\mathfrak{gl}(2m, \mathbb{R}), SO(2m)) \cong \bigwedge (y_1, y_3, \dots, y_{2m-1}, y_{2m}),$$

where y_{2k-1} are the same as above and $y_{2m} \in H^{2m}(\text{End}(V), \text{Sk}(V))$ is a nonzero class determined by the Pfaffian. For details concerning elements y_{2m} see [4]. Example 4.1 shows that if $\dim V$ is even, the homomorphism $\Delta_{(\text{End}(V), \text{Sk}(V))\#}$ is not a monomorphism.

Example 4.3 Let $\mathfrak{g}, \mathfrak{h}$ be Lie algebras and $\mathfrak{g} \oplus \mathfrak{h}$ their direct product. The characteristic homomorphism of the pair $(\mathfrak{g} \oplus \mathfrak{h}, \mathfrak{h})$ is a monomorphism. It is equal to

$$\Delta_{(\mathfrak{g} \oplus \mathfrak{h}, \mathfrak{h})\#} : H^\bullet(\mathfrak{g}) \rightarrow H^\bullet(\mathfrak{g}) \otimes H^\bullet(\mathfrak{h}), \quad \Delta_{\# \mathfrak{g} \oplus \mathfrak{h}, \mathfrak{h}}([\Phi]) = [(-1)^{|\Phi|} \cdot \Phi] \otimes 1.$$

4.2 The Exotic Universal Characteristic Homomorphism of Principal Fibre Subbundles

We recall briefly secondary (exotic) flat characteristic classes for flat principal bundles [13] and its connection to the exotic characteristic classes for a pair of a Lie algebroids of a suitable vector bundle and its reduction [4].

Let P be a G -principal fibre bundle on a smooth manifold M , $\omega \subset TP$ a flat connection in P and $P' \subset P$ a connected H -reduction, where $H \subset G$ is a closed Lie subgroup of G . Let us consider Lie algebroids $A(P)$, $A(P')$ of the principal bundles P, P' , respectively, the induced flat connection $\omega^A : TM \rightarrow A(P)$ in the Lie algebroid $A(P)$, and the secondary characteristic homomorphism

$$\Delta_{(A(P), A(P'), \omega^A)\#} : H^\bullet(\mathfrak{g}, A(P')) \longrightarrow H_{dR}^\bullet(M)$$

for the FS-Lie algebroid $(A(P), A(P'), \omega^A)$. Moreover, let

$$\Delta_{(P, P', \omega)\#} : H^\bullet(\mathfrak{g}, H) \longrightarrow H_{dR}^\bullet(M)$$

be the classical homomorphism on principal fibre bundles (F. Kamber, Ph. Tondeur [13]), where $H^\bullet(\mathfrak{g}, H)$ is the relative Lie algebra cohomology of (\mathfrak{g}, H) (see [13], [5]). There exists an isomorphism of algebras $\kappa : H^\bullet(\mathfrak{g}, H) \xrightarrow{\cong} H^\bullet(\mathfrak{g}, A(P'))$ such that

$$\Delta_{(A(P), A(P'), \omega^A)\#} \circ \kappa = \Delta_{(P, P', \omega)\#} \quad (4.1)$$

(see [19, Theorem 6.1]). Hence, their characteristic classes are identical. In [4] we showed that the homomorphism

$$\Delta_{(P, P')\#} := \Delta_{(A(P), A(P'))\#} \circ \kappa : H^\bullet(\mathfrak{g}, H) \longrightarrow H^\bullet(A(P)) \longrightarrow H_{dR}^{r\bullet}(P)$$

factorizes $\Delta_{(P, P', \omega)\#}$ for any flat connection ω in P , i.e. the following diagram commutes

$$\begin{array}{ccc} & H_{dR}^\bullet(P) & \\ \Delta_{(P, P')\#} \nearrow & & \searrow \omega^\# \\ H^\bullet(\mathfrak{g}, H) & \xrightarrow{\Delta_{(P, P', \omega)\#}} & H_{dR}^\bullet(M), \end{array}$$

where $\omega^\#$ on the level of right-invariant differential forms $\Omega^r(P)$ is given as the pullback of differential forms. In particular, if G is a compact, connected Lie group and P' is a connected H -reduction in a G -principal bundle P , $H \subset G$, then there exists a homomorphism of algebras

$$\Delta_{(P, P')\#} : H^\bullet(\mathfrak{g}, H) \longrightarrow H_{dR}^\bullet(P)$$

(called a *universal exotic characteristic homomorphism* for the pair $P' \subset P$) such that for arbitrary flat connection ω in P , the characteristic homomorphism $\Delta_{(P, P', \omega)\#} : H^\bullet(\mathfrak{g}, H) \rightarrow H_{dR}^\bullet(M)$ is factorized by $\Delta_{(P, P')\#}$, i.e. the following diagram is commutative

$$\begin{array}{ccc} & H_{dR}^\bullet(P) & \\ \Delta_{(P, P')\#} \nearrow & & \searrow \omega^\# \\ H^\bullet(\mathfrak{g}, H) & \xrightarrow{\Delta_{(P, P', \omega)\#}} & H_{dR}^\bullet(M). \end{array}$$

5 About a Monomorphicity of the Universal Exotic Characteristic Homomorphism for a Pair of Transitive Lie Algebroids

Consider a pair (A, B) of transitive Lie algebroids on a manifold M , $B \subset A$, $x \in M$, and a pair of adjoint Lie algebras $(\mathfrak{g}_x, \mathfrak{h}_x)$. Clearly, the inclusion $\iota_x : (\mathfrak{g}_x, \mathfrak{h}_x) \rightarrow (A, B)$ is a homomorphism of pairs of Lie algebroids over $\{*\} \hookrightarrow M$. Theorem 3.1 gives rise to the commutative diagram

$$\begin{array}{ccc} H^\bullet(\mathfrak{g}, B) & \xrightarrow{\Delta_{(A,B)\#}} & H^\bullet(A) \\ \downarrow \iota_x^{+\#} & & \downarrow \iota_x^\# \\ H^\bullet(\mathfrak{g}_x, \mathfrak{h}_x) & \xrightarrow{\Delta_{(\mathfrak{g}_x, \mathfrak{h}_x)\#}} & H^\bullet(\mathfrak{g}_x). \end{array} \quad (5.1)$$

Obviously, if the left and lower homomorphisms are monomorphisms, then $\Delta_{(A,B)\#}$ is a monomorphism as well. The homomorphism $\iota_x^{+\#}$ is a monomorphism, if each invariant element $v \in (\bigwedge (\mathfrak{g}_x/\mathfrak{h}_x)^*)^{\mathfrak{h}_x}$ can be extended to a global invariant cross-section of the vector bundle $\bigwedge (\mathfrak{g}/\mathfrak{h})^*$. In consequence, we obtain the following theorem linking the Koszul homomorphism with exotic characteristic classes:

Theorem 5.1 *Let (A, B) be a pair of transitive Lie algebroids over a manifold M , $B \subset A$, and let $(\mathfrak{g}_x, \mathfrak{h}_x)$ be a pair of adjoint Lie algebras at $x \in M$. If each element of $(\bigwedge (\mathfrak{g}_x/\mathfrak{h}_x)^*)^{\mathfrak{h}_x}$ can be extended to an invariant cross-section of $\bigwedge (\mathfrak{g}/\mathfrak{h})^*$ and the Koszul homomorphism $\Delta_{(\mathfrak{g}_x, \mathfrak{h}_x)\#}$ for the pair $(\mathfrak{g}_x, \mathfrak{h}_x)$ is a monomorphism, then $\Delta_{(A,B)\#}$ is a monomorphism.*

The assumptions of the above theorem hold for integrable Lie algebroids A and B ($B \subset A$), i.e. if $A = A(P)$ for some principal G -bundle P and $B = A(P')$ for some reduction P' of P with connected structural Lie group $H \subset G$ (remark: on account of Theorem 1.1 in [15] for any transitive Lie subalgebroid $B \subset A(P)$ there exists a connected reduction P' of P having B as its Lie algebroid, i.e. $B = A(P')$, but, in general, the structural Lie group of P' may be not connected). Let \mathfrak{g} and \mathfrak{h} denote the Lie algebras of G and H , respectively. The representation $\text{ad}_{B, \mathfrak{h}}$ is integrable: it is a differential of the representation $\text{Ad}_{P', \mathfrak{h}} : P' \rightarrow L(\mathfrak{g}/\mathfrak{h})$ of the principal fibre bundle P' defined by $z \mapsto [\hat{z}]$, see [19, p. 218]. We recall that for each $z \in P'$, the isomorphism $\hat{z} : \mathfrak{g} \rightarrow \mathfrak{g}_x$, $v \mapsto [(A_z)_{*e} v]$ ($A_z : G \rightarrow P$, $a \mapsto za$) maps \mathfrak{h} onto \mathfrak{h}_x (see [16, Sec. 5.1]) and determines an isomorphism $[\hat{z}] : \mathfrak{g}/\mathfrak{h} \rightarrow \mathfrak{g}_x/\mathfrak{h}_x$. Therefore (see also ([16, Prop. 5.5.2-3])), we have a natural isomorphism

$$\kappa : \left(\bigwedge (\mathfrak{g}_x/\mathfrak{h}_x)^* \right)^H \xrightarrow{\cong} \left(\Gamma \left(\bigwedge (\mathfrak{g}/\mathfrak{h})^* \right) \right)^{\Gamma(B)}$$

and because of the connectedness of H , $(\bigwedge (\mathfrak{g}_x/\mathfrak{h}_x)^*)^H = (\bigwedge (\mathfrak{g}_x/\mathfrak{h}_x)^*)^{\mathfrak{h}_x}$, which gives that $\iota_x^{+\#}$ is an isomorphism. In this way we obtain the following corollary:

Corollary 5.2 *Let (A, B) be a pair of Lie algebroids, $B \subset A$, where A is an integrable Lie algebroid via a principal fibre bundle P and let $(\mathfrak{g}_x, \mathfrak{h}_x)$ be a pair of adjoint Lie algebras at $x \in M$. If the structure Lie group of the connected reduction P' of P such that $A(P') = B$ is a connected Lie group and the Koszul homomorphism $\Delta_{(\mathfrak{g}_x, \mathfrak{h}_x)\#}$ for the pair $(\mathfrak{g}_x, \mathfrak{h}_x)$ is a monomorphism (for examples see Example 4.1), then $\Delta_{(A,B)\#}$ is a monomorphism as well.*

Let (A, B) be a pair of transitive Lie algebroids over a manifold M , $B \subset A$, for which the kernels \mathbf{g}, \mathbf{h} of anchors are abelian Lie algebra bundles, and let $x \in M$. An example of the mentioned Lie algebroid is the Lie algebroid $A(G, H)$ of a nonclosed and connected Lie subgroup H of G (see [16]). The existence of such nonintegrable Lie algebroids is shown in [1]. Since the homomorphism $H^\bullet(\mathbf{g}_x) = \bigwedge (\mathbf{g}_x)^* \rightarrow \bigwedge (\mathbf{h}_x)^* = H^\bullet(\mathbf{h}_x)$ induced by the inclusion $\mathbf{g}_x \hookrightarrow \mathbf{h}_x$ is surjective and $(\mathbf{g}_x, \mathbf{h}_x)$ is a reductive pair of Lie algebras, the Koszul homomorphism $\Delta_{(\mathbf{g}_x, \mathbf{h}_x)\#}$ is injective (see Example 4.1). Hence, in view of Theorem 5.1 we obtain that if (A, B) is a pair of transitive Lie algebroids on a manifold M such that kernels of their anchors \mathbf{g}, \mathbf{h} are abelian Lie algebra bundles and each element of $(\bigwedge (\mathbf{g}_x/\mathbf{h}_x)^*)^{\mathfrak{h}_x}$ can be extended to an invariant cross-section of $\bigwedge (\mathbf{g}/\mathbf{h})^*$ for all $x \in M$, then $\Delta_{(A, B)\#}$ is a monomorphism.

Remark 5.3 An example of a nontrivial universal characteristic class determined by the Pfaffian (see [4]) shows that there exists a pair of transitive and integrable Lie algebroids (A, B) for which the left arrow in the diagram (5.1) describes an isomorphism, the bottom one is not a monomorphism, however, the top one is a monomorphism.

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